INVERSE MATRIX EXAMPLE

Adjugate Formula for the Inverse Matrix

Let

$$A_{(n,n)} = (a_{ij})$$

be an $n \times n$ matrix with

$$\det(A) \neq 0,$$

and let

$$a^*_{ij} = (-1)^{i+j} |A_{ij}|,$$

where $|A_{ij}|$ denotes the determinant of the minor obtained by deleting the i-th row and the j-th column.

Then:

$$A^{-1} = \frac{1}{\det(A)} \left(a^*_{ij}\right)^T$$

This means that the inverse matrix A^{-1} exists if and only if

$$\det(A) \neq 0.$$

Such matrices are called **non-singular (regular)**.

If

$$\det(A) = 0,$$

then A is called a **singular matrix**.

Example

Find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}.$$

First, we calculate the determinant of A. (If it is zero, there is no need to continue.) As an exercise, we expand along the first column:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -8 + 12 = 4 \neq 0.$$

Next, we determine the cofactors k_{ij} (pay attention to the signs):

$$\begin{aligned} &k_{11} = -8, \quad &k_{12} = -2, \quad &k_{13} = 6, \\ &k_{21} = 6, &k_{22} = 1, &k_{23} = -3, \\ &k_{31} = -2, &k_{32} = 1, &k_{33} = 1. \end{aligned}$$

Finally, the **adjugate matrix** is obtained as the transpose of the cofactor matrix $K = (k_{ij})$. Apart from the factor

$$\frac{1}{\det(A)} = \frac{1}{4},$$

this is already the inverse:

$$A^{-1} = \frac{1}{4} \begin{pmatrix} -8 & 6 & -2 \\ -2 & 1 & 1 \\ 6 & -3 & 1 \end{pmatrix}.$$

(The matrix inside the brackets is the **adjugate matrix**.)