Rank of a Matrix

1. Definition

The **maximum number of linearly independent column vectors or row vectors** is called the **rank of the matrix**.

In a matrix, the largest number of linearly independent column vectors is always equal to the largest number of linearly independent row vectors.

Example 1

Given the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 4 \\ 3 & 5 & 6 \end{pmatrix}$$

Since the **third column is a multiple of the first column**, the three vectors are linearly dependent.

However, the first two columns are not multiples of one another and are therefore linearly independent. Hence, the rank of this matrix is 2:

$$\boxed{\operatorname{rank}(A) = 2}$$

2. Calculating the Rank of a Matrix

To determine the rank:

1. **Convert the matrix into row-echelon form**
2. **Write down the solution**

Step 1

For this purpose, we use the **Gaussian elimination algorithm**.

Step 2

The rank of a matrix corresponds to the **number of non-zero rows** of the matrix in row-echelon form.

Example

Given the matrix

$$A = \begin{pmatrix} 0 & -2 & 2 & 4 \\ 2 & -1 & -1 & 1 \\ 2 & -2 & 0 & 3 \end{pmatrix}$$

Calculate the rank of the matrix.

1. Convert the matrix into row-echelon form

First, **interchange row 1 and row 2**:

$$\begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 2 & -2 & 0 & 3 \end{pmatrix}$$

Then perform

$$R_3 \leftarrow R_3 - R_1$$

giving

$$\begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

Next,

$$R_3 \leftarrow R_3 - 0.5R_2$$

giving

$$\begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Write down the solution

There are **two rows that do not consist exclusively of zeros**.

Therefore,

$$\boxed{\text{rank}(A) = 2}$$

3. Special Case: Rank of Regular Matrices

The rank of a **regular matrix** corresponds to the number of rows or columns of the matrix.

Reminder: Regular matrices are matrices that are **invertible**.

A square matrix A is invertible if and only if

$$\det(A) \neq 0$$

Reminder: $\det(A)$ is the **determinant** of matrix A.

Example

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 4 & 4 \\ 1 & 4 & 7 \end{vmatrix} = -10$$

Since the determinant is **not equal to zero** and the square matrix has 3 rows and 3 columns, the matrix has rank 3.

$$\boxed{\text{rank}(A) = 3}$$

Example

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 4 & 4 \\ 3 & 5 & 6 \end{vmatrix} = 0$$

Since the determinant is **equal to zero**, this is a **singular matrix**, i.e. a matrix that is **not invertible**.

Regarding the rank of this matrix, we can only state that it is **less than 3**.

The exact rank can be calculated using one of the methods discussed above.

Important

The **rank of a matrix is important for determining the solvability of systems of linear equations**.