DETERMINANTS

Cofactor

Contents

1. Formula

$$A_{ij} = (-1)^{i+j} \cdot D_{ij}$$

Here, A_{ij} is the **cofactor**, which consists of the multiplication of a **sign factor** $(-1)^{i+j}$ and a **minor** D_{ij}.

1.1 Minor

D_{ij} is the **minor** obtained by deleting the i-th row and the j-th column.

In the following examples, we consider several minors of a 3×3 determinant.

Example 1

D_{11} is the minor obtained by deleting the **1st row and 1st column**:

$$D_{11} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Example 2

D_{13} is the minor obtained by deleting the **1st row and 3rd column**:

$$D_{13} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 3

D_{32} is the minor obtained by deleting the **3rd row and 2nd column**:

$$D_{32} = \begin{vmatrix} a_{11} & \cancel{a_{12}} & a_{13} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

1.2 Sign Factor

The sign factor

$$(-1)^{i+j}$$

assigns a sign to every minor.

Here, i is the **row index** and j is the **column index**.

- If $i + j$ is **even**, the sign is **positive.**
- If $i + j$ is **odd**, the sign is **negative.**

Remember:
The element in the **upper-left corner** is assigned a **plus sign.**

Application Using the Laplace Expansion Theorem

The **Laplace expansion theorem** is based on reducing a determinant to successively smaller determinants.

For example, a 4×4 determinant can be reduced to a 3×3 determinant. This 3×3 determinant can then be reduced using the expansion theorem to an even smaller determinant — a 2×2 determinant — or calculated using another, simpler formula. Determinats Part 3 deutsch

Expansion along the i-th row

$$|A| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} D_{ij}$$

Expansion along the j-th column

$$|A| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} D_{ij}$$

where:

- $|A|$ is the determinant of matrix A
- a_{ij} is the **element at the intersection of row i and column j**
- $(-1)^{i+j}$ is the **sign factor**
- D_{ij} is the minor obtained by deleting the i-th row and j-th column.

Example

Calculate the determinant of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{pmatrix}$$

using the **Laplace expansion theorem.**

1st column

For the first element:

$$a_{11}(-1)^{1+1}D_{11} = 1 \cdot (-1)^{1+1} \begin{vmatrix} 6 & 5 \\ 9 & 8 \end{vmatrix}$$

- a_{11}: element at the intersection of the 1st row and 1st column
- $(-1)^{1+1}$: sign factor, which is positive because the exponent is even
- D_{11}: minor obtained by deleting the 1st row and 1st column.

2nd column

$$a_{12}(-1)^{1+2}D_{12} = 3 \cdot (-1)^{1+2} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

- a_{12}: element at the intersection of the 1st row and 2nd column
- $(-1)^{1+2}$: sign factor, which is negative because the exponent is odd
- D_{12}: minor obtained by deleting the 1st row and 2nd column.

3rd column

$$a_{13}(-1)^{1+3}D_{13} = 2 \cdot (-1)^{1+3} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$

- a_{13}: element at the intersection of the 1st row and 3rd column
- $(-1)^{1+3}$: sign factor, which is positive because the exponent is even
- D_{13}: minor obtained by deleting the 1st row and 3rd column.

Calculate the result

$$|A| = 1(-1)^{1+1} \begin{vmatrix} 6 & 5 \\ 9 & 8 \end{vmatrix} + 3(-1)^{1+2} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} + 2(-1)^{1+3} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$

$$= 1(6 \cdot 8 - 9 \cdot 5) - 3(4 \cdot 8 - 7 \cdot 5) + 2(4 \cdot 9 - 7 \cdot 6)$$

$$= 1(48 - 45) - 3(32 - 35) + 2(36 - 42)$$

$$= 1 \cdot 3 - 3(-3) + 2(-6)$$

$$= 3 + 9 - 12$$

$$\boxed{|A| = 0}$$