4. h-Method

We are already familiar with the diagram from the chapter on **difference quotients**.

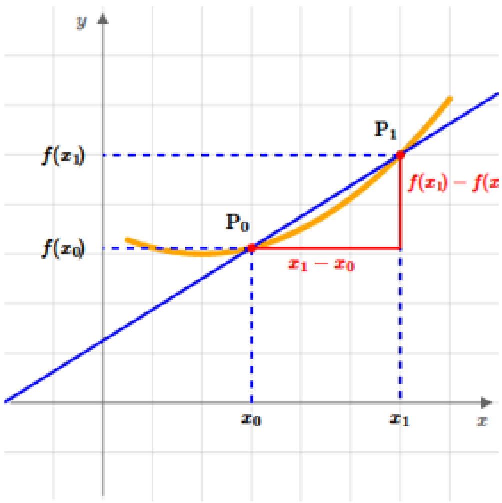
The difference quotient is known to be:

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The idea behind the **h-method** is that instead of substituting a specific point x_1 into the difference quotient, we use a placeholder h, for which:

$$h = x_1 - x_0$$

The variable h—hence the name **h-method**—therefore represents the distance between two x-values.



We therefore change the notation of the difference quotient so that:

$$h = x_1 - x_0$$

We solve the equation for x_1:

$$x_1 = x_0 + h$$

Consequently:

$$f(x_1) = f(x_0 + h)$$

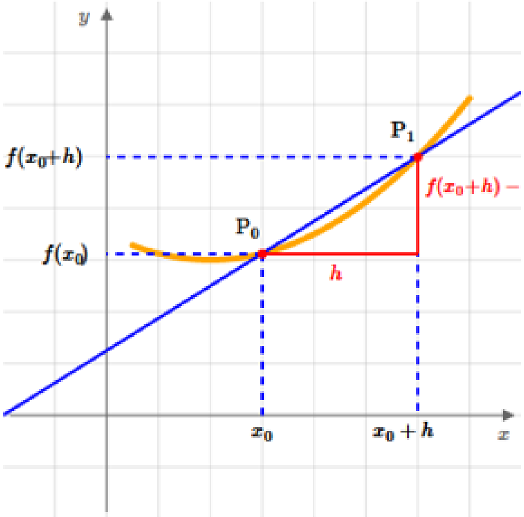
The difference quotient as a function of h is therefore:

$$m = \frac{f(x_0 + h) - f(x_0)}{h}$$

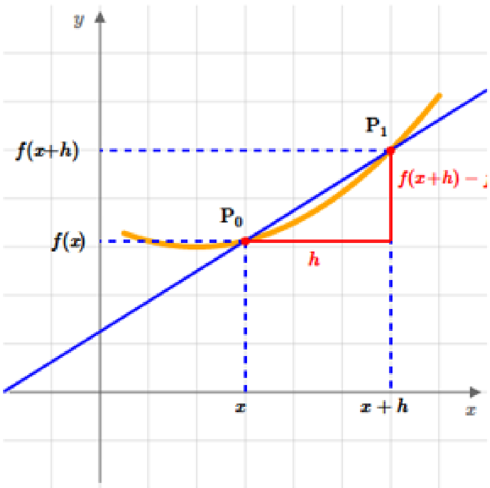
Since x_1 no longer appears in the formula above, we can simply write x instead of x_0.

The difference quotient in the new notation is:

$$m = \frac{f(x+h) - f(x)}{h}$$



somit:



Up to this point, we have expressed the difference quotient only in terms of the variable h. However, what we are looking for is the **derivative function**—that is, the function that assigns the value of the differential quotient to every point x_0 (or simply x).

From the chapter on differential quotients, we know:

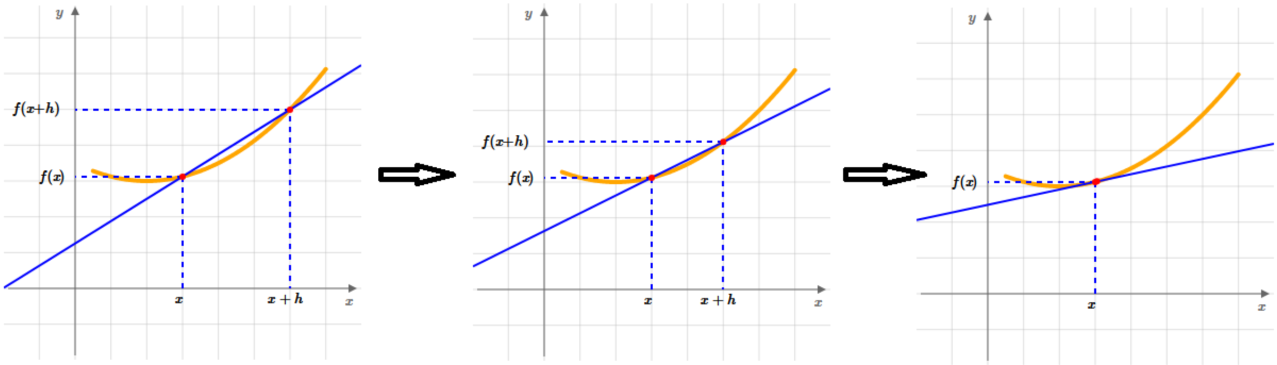
> **The differential quotient is the limit of the difference quotient.**

In this case, the limit means that h approaches 0.

Therefore, the differential quotient as a function of h is:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The animation clearly shows what happens graphically when h approaches 0: the **secant line becomes a tangent line.** We already know this from the previous chapter.



By rewriting the differential quotient

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

in terms of the variable h, with

$$h = x_1 - x_0,$$

we obtain:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Using this formula, we can now calculate a function that assigns the value of the differential quotient to every point x. This is the required **derivative function** $f'(x)$.

In summary:

> **The h-method is a procedure for deriving derivative functions.**

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)}$$

$f(x+h)$ means that in the function $f(x)$, we simply substitute $x + h$ for x.

For example, if

$$f(x) = x^2,$$

then:

$$f(x+h) = (x+h)^2.$$

Let´s have a look

Example

Calculate the derivative of the function

$$f(x) = x^2$$

using the h-method.

1. Write down the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Substitute the values

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

3. Simplify the expression

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \qquad | \text{ Expand}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \qquad | \text{ Combine like terms}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} \qquad | \text{ Factor out}$$

$$= \lim_{h \to 0} \frac{h(2x+h)}{h} \qquad | \text{ Cancel}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x+h)$$

4. Calculate the limit

$$= 2x + 0$$

$$= 2x$$

Therefore, the derivative of the function

$$f(x) = x^2$$

is:

$$\boxed{f'(x) = 2x}$$