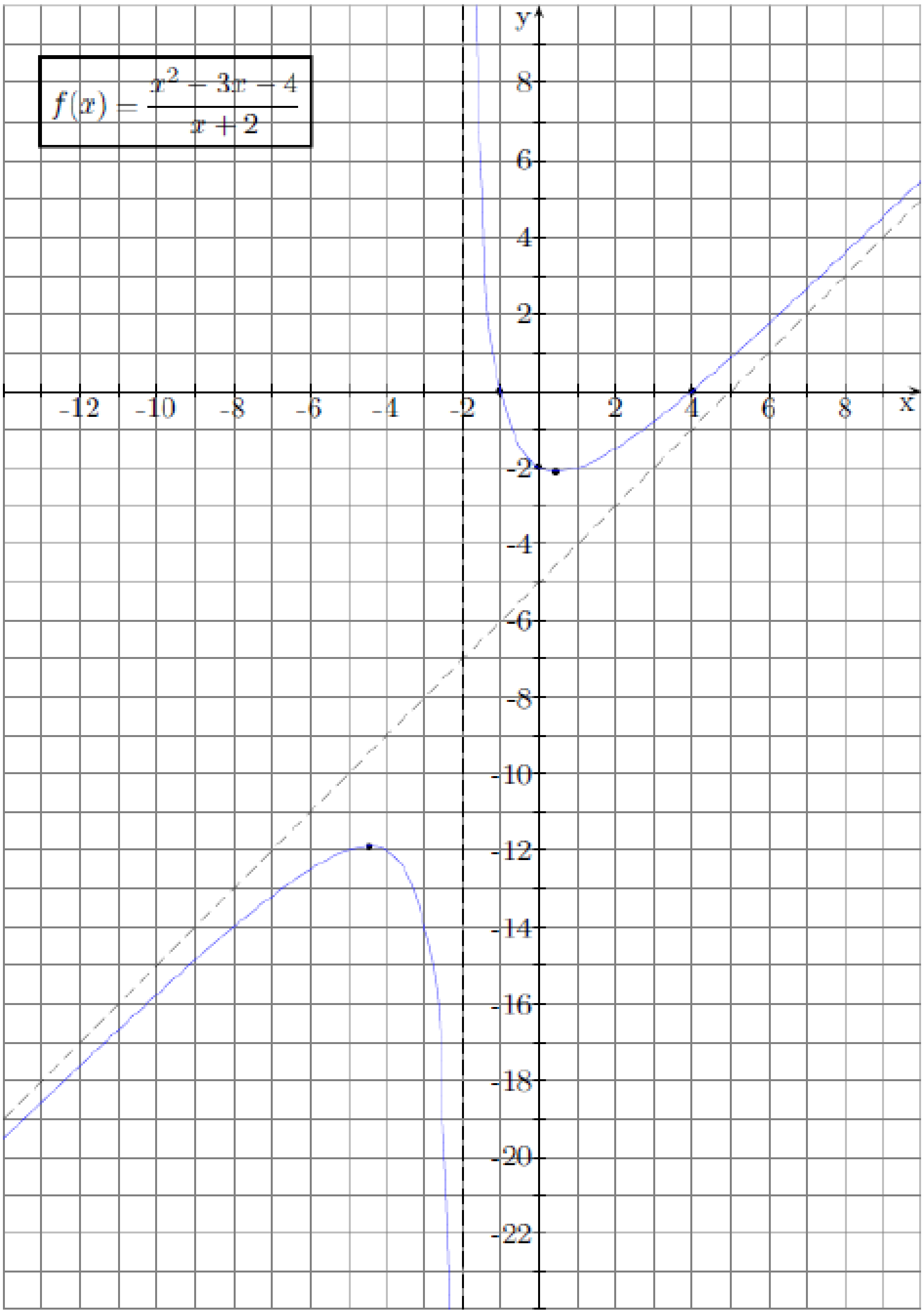
COMPLETE CURVE DISCUSSION OF A RATIONAL FUNCTION

Example of a complete curve discussion of a rational function

Given the function:

$$f(x) = \frac{x^2 - 3x - 4}{x + 2}$$



a) Domain

It can be seen that the denominator becomes zero at $x = -2$.
The numerator is not zero at this value of x.
Therefore, the domain is:

$$D = \{x \in \mathbb{R} \mid x \neq -2\}$$

Other notations are also possible, e.g.:

$$D = \{x \in \mathbb{R};\, x \neq -2\}$$

The excluded point in the domain represents a **pole**.

Vertical asymptote:

$$\boxed{x = -2 \quad \text{(odd-order pole)}}$$

This is defined as a pole with a **change of sign**.

We approach -2 from the left, i.e. from values smaller than -2:

$$\lim_{x \to -2^-} f(x) = -\infty$$

We approach -2 from the right, i.e. from values greater than -2:

$$\lim_{x \to -2^+} f(x) = +\infty$$

Approching left side

$$\lim_{x \overset{<}{\to} -2} f(x) \quad = \quad \lim_{x \overset{<}{\to} -2} \frac{\overbrace{x^2 - 3x - 4}^{\to 6}}{\underbrace{x + 2}_{\overset{<}{\to} 0}} = -\infty$$

Approching right side

$$\lim_{x \overset{>}{\to} -2} f(x) \quad = \quad \lim_{x \overset{>}{\to} -2} \frac{\overbrace{x^2 - 3x - 4}^{\to 6}}{\underbrace{x + 2}_{\overset{>}{\to} 0}} = +\infty$$

b) Intersections with the coordinate axes

To find the zeros of the function f, and therefore the intersections with the **x-axis**, the numerator must be set equal to zero:

$$x^2 - 3x - 4 = 0$$

For such quadratic equations, the familiar **quadratic formula** or p/q **formula** is routinely used (see formula sheet):

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

A more elegant method, however, is to factor the expression into linear factors:

$$x^2 - 3x - 4 = (x+1)(x-4)$$

The intersection points with the x-axis can therefore be read directly:

$$S_{x1} = (-1|0)$$

$$S_{x2} = (4|0)$$

The intersection with the **y-axis** is calculated by evaluating the function at $x = 0$:

$$f(0) = -2$$

Therefore:

$$S_y = (0|-2)$$

c) Critical Points / Extrema

Definition

Critical points (extrema) of a function are points where the slope of the tangent to the graph is zero. In other words, the tangent is parallel to the x-axis.

The slope of a function at an arbitrary point is determined using the **first derivative**:

$$f'(x_E) = 0$$

To determine mathematically whether a critical point is a **local minimum or local maximum**, we examine the curvature of the function at the extremum.

For the curvature convention used in the document:

$$f''(x_E) < 0 \quad \Rightarrow \quad \text{local maximum}$$

$$f''(x_E) > 0 \quad \Rightarrow \quad \text{local minimum}$$

Special case

$$f''(x_E) = 0 \quad \Rightarrow \quad \text{saddle point}$$

This special case is treated separately.

Derivative of the function

Applying the **quotient rule**:

$$f'(x) = \frac{(x+2)(2x-3) - (x^2 - 3x - 4) \cdot 1}{(x+2)^2}$$

After simplification:

$$\boxed{f'(x) = \frac{x^2 + 4x - 2}{(x+2)^2}}$$

Setting the first derivative equal to zero gives the critical points:

$$x_{E1} = -\sqrt{6} - 2$$

$$x_{E2} = +\sqrt{6} - 2$$

The coordinates of the extrema are therefore:

$$E_1 = (x_{E1} | f(x_{E1}))$$

$$E_2 = (x_{E2} | f(x_{E2}))$$

Since the second derivative will also be needed later, the quotient rule is applied again.

$$\boxed{f''(x) = \frac{12}{(x+2)^3}}$$

d) Points of Inflection

Definition

Points of inflection are points where the slope of the function is greatest (whether positive or negative).

In other words, the **second derivative** of a function represents the extrema of the first derivative.

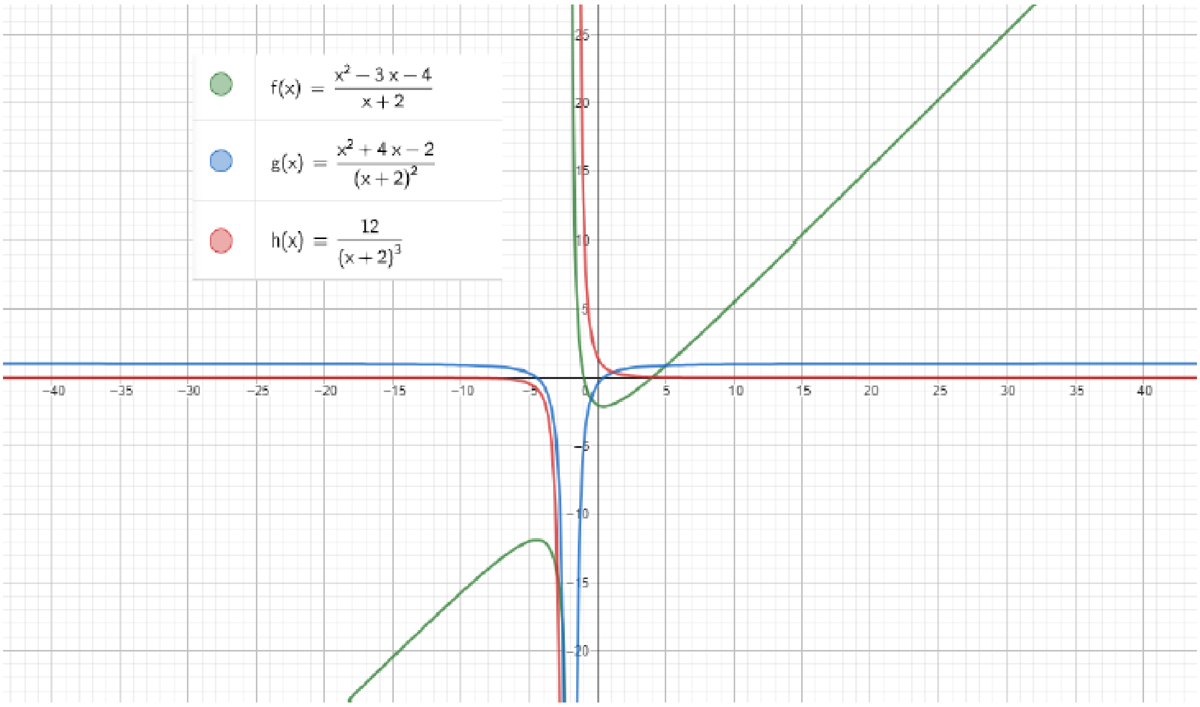
Graphical illustration

$$g(x) = f'(x)$$

$$h(x) = f''(x)$$

The first derivative (blue graph) has two zeros → **two extrema.**

The second derivative (red graph) has no zeros → **no point of inflection**.



$$g(x) = f'(x)$$
$$h(x) = f''(x)$$

e) Monotonicity

Since the critical points have been calculated, we can create a **monotonicity table**.

Interval	$f'(x)$	$f(x)$
$x < -\sqrt{6} - 2$	> 0	increasing
$-\sqrt{6} - 2 < x < -2$	< 0	decreasing
$-2 < x < \sqrt{6} - 2$	< 0	decreasing
$x > \sqrt{6} - 2$	> 0	increasing

f) Concavity

As can be seen from the numerator, the second derivative

$$f''(x)$$

cannot become zero.

Therefore, the sign of $f''(x)$ can change at most at the excluded point $x = -2$.

	$x < -2$	$x > -2$
$f''(x)$	< 0	> 0
$f(x)$	concave down	concave up

Therefore, the table for the concavity behavior contains only two intervals.

g) Oblique (Slant) Asymptote

Since the degree of the numerator is one higher than the degree of the denominator, we can expect an **oblique asymptote.**

This is calculated using **polynomial long division**:

$$(x^2 - 3x - 4) : (x + 2) = x - 5 + \frac{6}{x + 2}$$

For

$$x \to \infty$$

we can therefore use the oblique asymptote:

$$\boxed{y = x - 5}$$

h) Point Symmetry about a Point $(x_0|y_0)$

At first glance, no symmetry is apparent.

Since the limits for

$$x \to -\infty$$

and

$$x \to +\infty$$

are different, only **point symmetry** can be considered. This is possible only if the center of symmetry lies simultaneously on both asymptotes:

$$x = -2$$

and

$$y = x - 5$$

The general formula for point symmetry about the origin is assumed to be known:

$$f(-x) = -f(x)$$

First, we determine the intersection of the two asymptotes.

Substituting $x = -2$ into

$$y = x - 5$$

gives:

$$y = -2 - 5 = -7$$

Therefore, the intersection point of the asymptotes is:

$$\boxed{(-2|-7)}$$

The general condition for point symmetry with respect to $(x_0|y_0)$ is:

$$\boxed{f(x_0 - h) + f(x_0 + h) = 2y_0}$$

Thus, the graph has point symmetry about the point:

$$\boxed{(-2|-7)}$$

Range of the Function

Only the **range** remains to be determined.

Looking at the graph of the function, we observe that the function is not defined between the local maximum and the local minimum.

In section c), the critical points were calculated:

$$x_{E1} = -\sqrt{6} - 2$$
$$x_{E2} = +\sqrt{6} - 2$$

Using the second derivative, we can determine whether these points are a local minimum or maximum:

$$f'' > 0 \quad \Rightarrow \quad \text{minimum}$$
$$f'' < 0 \quad \Rightarrow \quad \text{maximum}$$

The corresponding function values are:

$$E_1 = \left(-\sqrt{6} - 2 \mid -2\sqrt{6} - 7\right)$$
$$E_2 = \left(+\sqrt{6} - 2 \mid 2\sqrt{6} - 7\right)$$

Finally, we obtain the range:

$$\boxed{W = \{y \in \mathbb{R} \mid -2\sqrt{6} - 7 \geq y \geq 2\sqrt{6} - 7\}}$$