

DIFFERENTIAL CALCULUS

Determining Limits

Most Important Methods for Determining Limits

- **Direct substitution:** Substitute the required value for the variable x (or n). If you obtain an ordinary number, you are done.
- **Indeterminate expressions ($0/0$ or ∞/∞):** If direct substitution produces an unclear expression, the term must be transformed, e.g. by factoring, canceling, or using binomial formulas.
- **L'Hôpital's Rule:** For fractions of the form

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty},$$

you may differentiate the numerator and denominator separately in order to determine the limit.

- **Behavior at infinity:** For fractions with $x \rightarrow \infty$, divide all terms by the highest power of the denominator. This eliminates terms that tend toward zero.

L'Hôpital's Rule

When differential calculus is known, we use **L'Hôpital's Rule**.

Application

L'Hôpital's Rule is used when determining the limit of a function of the form

$$f(x) = \frac{g(x)}{h(x)}$$

and the result is an indeterminate expression such as

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Theorem

L'Hôpital's Rule

$$\boxed{\lim_{x \rightarrow x_0} \frac{g(x)}{h(x)} = \lim_{x \rightarrow x_0} \frac{g'(x)}{h'(x)}}$$

L'Hôpital's Rule also applies when the limiting process is of the form

$$x \rightarrow +\infty$$

or

$$x \rightarrow -\infty.$$

Instructions

1. Differentiate the numerator function $g(x)$ and the denominator function $h(x)$ separately.
2. Calculate the limit of

$$\frac{g'(x)}{h'(x)}.$$

Notes

- There are cases in which the repeated application of this limit rule is necessary to reach the desired result.
- The rule may fail to provide a solution. Not every limit can be calculated using L'Hôpital's Rule.

Examples

Example 1

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \frac{0}{0}$$

Because of the indeterminate expression

$$\frac{0}{0},$$

we can apply L'Hôpital's Rule. In other words, we differentiate the numerator and denominator separately and then calculate the limit of the new expression:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{(e^x - 1)'}{(x)'} = \lim_{x \rightarrow 0} \frac{e^x}{1} = \lim_{x \rightarrow 0} e^x = 1.$$

Example 2

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

Because of the indeterminate expression

$$\frac{\infty}{\infty},$$

we can apply L'Hôpital's Rule. We differentiate the numerator and denominator separately and then calculate the limit of the new expression:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{(\ln x)'}{(x)'} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Transforming Indeterminate Expressions

L'Hôpital's Rule is valid only for indeterminate expressions of the form

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

The following other expressions:

- $0 \cdot \infty \quad \text{or} \quad \infty \cdot 0$
- $\infty - \infty$
- $0^0, \quad \infty^0, \quad 1^\infty$

can, however, be transformed using so-called **elementary transformations** so that L'Hôpital's Rule can subsequently be applied.

The following table shows the respective function, its limit, and the formula for the elementary transformation required in order to apply L'Hôpital's Rule.

Function $f(x)$	Limit	Elementary transformation
$g(x) \cdot h(x)$	$0 \cdot \infty$ or $\infty \cdot 0$	$\frac{g(x)}{1/h(x)}$ or $\frac{h(x)}{1/g(x)}$
$g(x) - h(x)$	$\infty - \infty$	$\frac{1/h(x) - 1/g(x)}{1/[g(x)h(x)]}$
$g(x)^{h(x)}$	$0^0, \infty^0, 1^\infty$	$e^{h(x) \ln(g(x))}$